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Criticality of K-contact vector fields

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Abstract

We prove that the characteristic vector field of a contact metric structure determines a contact invariant embedding or a J-holomorphic map into the tangent unit sphere bundle if and only if the contact form is K-contact. As a consequence, K-contact vector fields are minimal and harmonic sections. © 2002 Elsevier Science B.V. All rights reserved.

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1. Some minimal submanifolds theory

For a smooth immersion $f : M \to (N, g)$ of a manifold M into a Riemannian manifold N, the second fundamental form B is a normal bundle valued symmetric bilinear form on M defined for any two tangent vector fields X, Y by $B(X, Y) = (\nabla_X Y)^\perp$, where $X^\perp$ denotes the component of X normal to M in N, and ∇ is the Levi-Civita connection determined by g. The mean curvature vector field H of the immersion f is the trace of the second fundamental form B, i.e. for an orthonormal basis $\{E_i\}_{i=1,\dots,m}$ on M, $H = \sum_{i=1}^m B(E_i, E_i)$.

The mean curvature vector field H is the gradient of the volume functional, defined for compact M by

$$V(f) = \int_M f^* \Omega,$$

where Ω denotes the Riemannian volume element on N or equivalently, the Riemannian volume element of M in the induced metric $f^* g$. An immersion f_0 is a critical point of the volume functional, or simply a minimal immersion if and only if $H = 0$. Regardless of M

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being compact or not, an immersion $f : M \to (N, g)$ satisfying the above criticality condition will be called a minimal immersion. In this paper, we are specially interested in those immersions determined by unit vector fields. More precisely, let (M, g) be a Riemannian manifold and let Z be a unit vector field on M. Then $Z : M \to T^1M$ is an embedding of M into its tangent unit sphere bundle T^1M. The tangent unit sphere bundle will be endowed with the restriction of the Sasaki metric determined by g and the Levi-Civita connection map $\kappa : TTM \to TM$. We recall that the Sasaki metric $G(.,.)$ on TM is defined as follows:

$$G(X, Y) = g(\pi_* X, \pi_* Y) + g(\kappa X, \kappa Y), \tag{1}$$

where $\pi_* : TTM \to TM$ is the projection map [14].

A unit vector field is said to be minimal if it determines a minimal embedding into the tangent unit sphere bundle T^1M endowed with the restriction of the Sasaki metric also denoted by G. Parallel unit vector fields, when they exist, are absolute minimizers for the volume functional. It is natural to regard these as the "visually" best orginized unit vector fields and one hopes that visually better orginized vector fields are rewarded with minimum possible volume. On a space like the three-sphere, there are no parallel vector fields, so one looks instead for the next best thing: critical unit vector fields; after all, these are minimizers for some restricted variational problem.

2. Preliminaries on contact geometry

A contact form on a $2n+1$-dimensional manifold M is a 1-form α such that the identity $\alpha \wedge (\mathrm{d}\alpha)^n \neq 0$ holds everywhere on M. Given such a 1-form, there is always a unique vector field Z satisfying $\alpha(Z) = 1$ and $i_Z\,\mathrm{d}\alpha = 0$. The vector field Z is called the characteristic vector field of the contact form α and the corresponding one-dimensional foliation is called a contact flow.

Also, the contact manifold (M, α) admits a nonunique Riemannian metric g and a (1,1) tensor field J such that the following identities hold [1]:

$$JZ = 0, \qquad \alpha(Z) = 1, \qquad J^2 = -I + \alpha \otimes Z, \qquad \alpha(X) = g(Z, X), \tag{2}$$

$$g(JX, JY) = g(X, Y) - \alpha(X)\alpha(Y), \qquad g(X, JY) = \mathrm{d}\alpha(X, Y). \tag{3}$$

The tensors fields g, α, Z and J will be referred to as structure tensors and g is called a contact metric adapted to α. When the characteristic vector field Z is Killing with respect to a contact metric g, then the contact form is said to be K-contact. It is said to be Sasakian if a certain integrability condition is satisfied (see [1] for further details about these structures). We should however mention a few other facts that will be needed in our argument. Mainly, on any contact metric manifold, there is defined a symmetric $(1, 1)$ tensor field $h = \frac{1}{2} L_Z J$ which anticommutes with J and thus has trace 0. The identity

$$\nabla_X Z = -JX - JhX \tag{4}$$

is valid on any contact metric structure and a contact form is K-contact if and only if its corresponding tensor field h is identically zero, in which case the above identity reduces to

$$\nabla_X Z = -JX.$$

Contact geometry provides us with a rich collection of minimal submanifolds and minimal unit vector fields and it is our goal to present some of these in this note. To that end, we will need the notion of *contact invariant submanifold*.

Definition 1. A submanifold M in a contact manifold (N, α, J, Z) is said to be contact invariant if the characteristic vector field Z is tangent to M and JX is tangent to M whenever X is.

Clearly, a contact invariant submanifold inherits a contact metric structure from the ambient manifold. In this context, the inclusion of a contact invariant submanifold is a J-holomorphic map in the sense of [5]. It is also known that an isometric immersion is minimal if and only if it is harmonic. Since Ianus and Pastore [5, Theorem 2.2] have shown that J-holomorphic maps between contact metric manifolds are harmonic, Proposition 1 below is valid.

Proposition 1. *Let M be a contact metric invariant submanifold in a contact metric manifold* (N, α, Z, g, J). *Then M is minimal.*

Proposition 1 combined with Theorem 1 in this paper leads to a generalization of the following earlier result [4,8].

Proposition 2. *Let* (M, α, Z, g, J) *be a Sasakian structure on a closed manifold M. Then the characteristic vector field Z is minimal.*

Outside of the realm of Killing vector fields, we have the following result [8].

Proposition 3. *Let* (M, α, Z, g, J) *be a flat contact metric structure on a closed (necessarily three-dimensional) manifold M. Then the characteristic vector field Z is minimal.*

Remark 1. The converse of Proposition 1 is not true. Consider the standard flat contact metric structure on the three-torus $\mathbf{T}^3$ with characteristic vector field Z [10]. By Proposition 3 above, $Z : \mathbf{T}^3 \to T^1\mathbf{T}^3$, is minimal (critical for the volume functional). But, as a consequence of Theorem 1 in this paper, Z does not determine any invariant contact metric structure on $\mathbf{T}^3$ since no torus carries a K-contact structure [11].

3. The contact metric geometry of the tangent unit sphere bundle

The fundamental 1-form Θ is defined on T^*M by $\Theta_\mu(v) = \mu(\pi_* v)$, where $v \in T_\mu T^*M$ and $\pi_* : TT^*M \to TM$ is the differential of the projection map $\pi : T^*M \to M$. The 2-form $\Omega = -\mathrm{d}\Theta$ is a symplectic form on T^*M. We refer to [6] for the basics of symplectic geometry. Given a Riemannian metric g on M, the fundamental 1-form Θ pulls back to a 1-form $\tilde{\Theta}$ on TM, $\tilde{\Theta} = \flat^*\Theta$, where $\flat : TM \to T^*M$ is the usual musical isomorphism determined by the metric g. The 2-form $\tilde{\Omega} = -\mathrm{d}\tilde{\Theta} = \flat^*\,\mathrm{d}\Theta$ is a symplectic form on TM [7, pp. 246–247]. If we denote by $\kappa : TTM \to TM$ the Levi-Civita connection map, then

$$(\pi_*, \kappa) : TTM \to TM \oplus TM \tag{5}$$

is a vector bundle isomorphism along $\pi : TM \to M$. It determines an almost complex structure J_{TM} on TM such that if $(\pi_*, \kappa)(x) = (u, v)$, then $(\pi_*, \kappa)(J_{TM}x) = (-v, u)$ [2,12].

The tangent bundle of a Riemannian manifold M carries a distinguished vector field S called the geodesic spray. S is determined by $\pi_* S(p, v) = v$, and $\kappa S(p, v) = 0$ for any $(p, v) \in TM$. With respect to the Sasaki metric $G(.,.)$ defined in (1), the 1-form $\tilde{\Theta}$ satisfies the identity $\tilde{\Theta}(V) = G(S, V)$ for any section V of TTM, i.e., $\tilde{\Theta}$ and S are (Sasaki) metric duals. The symplectic form $\tilde{\Omega} = -\mathrm{d}\tilde{\Theta}$ is compatible with the pair (G, J_{TM}) in the sense that the identity

$$\tilde{\Omega}(X, Y) = G(J_{TM}X, J_{TM}Y)$$

holds for any pair (X, Y) of tangent vector fields on TM.

Letting $j : T^1M \to TM$ be the inclusion of the tangent unit sphere bundle as a hypersurface in TM, then the pulled back 1-form $\tilde{\alpha}_g = j^*\tilde{\Theta}$ is a contact form on T^1M [1] whose characteristic vector field is known as the geodesic flow of M. The kernel of $\tilde{\alpha}_g$ has associated almost complex operator J_{T^1M} determined by the equations

$$J_{T^1M}(S) = 0, \qquad J_{T^1M}(X) = J_{TM}X$$

for any X tangent to T^1M satisfying the identity $G(X, S) = 0$.

Under the identification (5), a vector tangent to TM at $(p, v) \in TM$ is a couple $(u, \nabla_u W)$, where $u \in T_pM$ and W is a vector field on M such that $W(p) = v$. A vector tangent to T^1M at $(p, v) \in T^1M$ is a couple $(u, \nabla_u W)$, where as above, u is a tangent vector at p, but now W is a unit vector field such that $W(p) = v$. Note that in this case, one has automatically the identity $g(v, \nabla_u W) = 0$. Now, if (M, α, Z, g, J) is a contact metric structure, a vector tangent to $Z(M) \subset T^1M$ at $(p, Z) \in Z(M)$ is a couple $(u, \nabla_u Z)$, where u is a tangent vector at $p \in M$. We also point out that under the identification (5), the geodesic flow on T^1M is given by $S(p, v) = (v, 0)$. From now on, it will be understood that the tangent unit sphere bundle is endowed with the restriction of the Sasaki metric (1).

Theorem 1. *Let (α, Z, g, J) be contact metric structure tensors on a manifold M. Then the sectional image $Z(M)$ is a contact invariant submanifold of T^1M if and only if α is a K-contact form.*

Proof. Clearly, the geodesic flow S is tangent to $Z(M)$. Indeed, at (p, Z) in $Z(M)$,

$$S(p, Z) = (Z, 0) = (Z, \nabla_Z Z) \in T_{(p,Z)}Z(M).$$

So we need only to show that the J_{T^1M} invariance of $Z(M)$ is equivalent to the K-contactness of α.

Let $(u, \nabla_u Z)$ be a tangent vector at $(p, Z) \in Z(M)$ such that $G((u, \nabla_u Z), (Z, 0)) = 0$, i.e., a tangent vector in the kernel distribution of the contact form $\tilde{\alpha}_g$ on T^1M. Then by definition and identity (4),

$$J_{T^1M}(u, \nabla_u Z) = (-\nabla_u Z, u) = (Ju + Jhu, u).$$

Therefore, following the description of a vector tangent to $Z(M)$, we see that $J_{T^1M}(u, \nabla_u Z)$ will be tangent to $Z(M)$ if and only if $\nabla_{Ju+Jhu} Z = u$. But a quick calculation using identity (4) again shows that $\nabla_{Ju+Jhu} Z = u - h^2 u$. Therefore, since h has trace 0, $Z(M)$ will be contact invariant if and only if $h = 0$, i.e. if and only if α is a K-contact form. □

From Proposition 1 and Theorem 1 above, we derive the following corollary.

Corollary 1. *Let (M, α, Z, g, J) be a K-contact structure. Then the characteristic vector field Z is minimal.*

This is a generalization of Proposition 2 to K-contact and to not necessarily compact manifolds.

4. Harmonic sections in contact geometry

K-contact vector fields turn out to be also interesting from the harmonic maps point of view. A harmonic map $f_0 : (M, g_M) \to (N, g_N)$ between two Riemannian manifolds is a critical point of the energy functional $\mathcal{E}(f) = \int_M \| f_* E_i \|^2 \Omega_M$, where $\{E_i\}_{i=1,\dots,m}$ is any orthonormal tangent frame on M and Ω_M denotes the Riemannian volume element on M [3]. In general, unless it is parallel, a unit vector field does not determine an isometric embedding into the tangent unit sphere bundle (with the restriction of the Sasaki metric). However, in the contact metric situation, we have the following refinement of Theorem 1.

Theorem 2. *Let (M, α, Z, g, J) be a contact metric structure. Then the characteristic vector field Z determines a J-holomorphic map between (M, J) and (T^1M, J_{T^1M}) if and only if the contact form α is K-contact.*

Proof. With the identification in (5) and the same notations as in the previous sections, it is clear that $Z_*(Z) = (Z, 0) = S(p, Z)$, i.e., as a map between two contact manifolds, Z exchanges the two characteristic vector fields involved. We need only to show that Z also exchange the complex operators J and J_{T^1M}.

Let $v \in T_pM$ be a tangent vector such that $\alpha(v) = 0$. On one hand,

$$J_{T^1M} Z_*(v) = J_{T^1M}(\pi_* Z_*(v), \kappa Z_* v) = J_{T^1M}(v, \nabla_v Z) = (Jv + Jhv, v). \quad (6)$$

On the other hand,

$$Z_*(Jv) = (Jv, \nabla_{Jv} Z) = (Jv, v - hv). \quad (7)$$

Identities (6) and (7) show that Z is a J-holomorphic map if and only if $h = 0$, i.e. if and only if α is K-contact. □

A unit vector field which determines a harmonic map into the tangent unit sphere bundle is called a harmonic section [13]. Since Ianus and Pastore [5] have shown that J-holomorphic maps between contact metric manifolds are harmonic, Theorem 2 above implies the following corollary.

Corollary 2. *Let* (M, α, Z, g, J) *be a* K*-contact metric structure. Then, the characteristic vector field Z is a harmonic section*.

Remark 2. Proposition 2 has a harmonic section analog. Mainly, the characteristic vector field of a Sasakian structure on a closed manifold is a harmonic section [9]. In this context, Corollary 2 is a generalization of this fact to K-contact and to not necessarily compact manifolds.

From the energy point of view, Proposition 3 has the following counterpart [9].

Proposition 4. *The characteristic vector field of a flat contact metric structure is a (unstable) harmonic section*.

Propositions 3 and 4 lead us to examples of critical unit contact vector fields which neither determine any contact invariant submanifolds nor determine any J-holomorphic embeddings in the tangent unit sphere bundle. Thus our "contact invariant" and "J-holomorphic embedding" conditions are only sufficient, not necessary for criticality.

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